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Preliminary Studies of Formation and Stability of Concentrated Vortices Near the Ground

by

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I. INTRODUCTION

This report summarizes the present status of a continuing investigation into the mysteries of the small-scale atmospheric vortex, usually known as the "dust devil" or "whirlwind." Many of the basic concepts of vortex flow presented below apply to the larger-scale tornado as well. The reference literature on the theory of small-scale vortices in meteorological journals is relatively sparse; however, there is an extensive literature on basic vortex motions available in journals on fluid mechanics.

The first application of the Navier-Stokes equations of fluid motion to a viscous vortex problem seems to have been the investigations of Taylor (1918) and Terazawa (1922), summarized in Dryden (1956), on the decay of a vortex consisting only of tangential velocities. Burgers (1940-1948), Rott (1958-1959), Sullivan (1959), Donaldson and Sullivan (1960), Long (1958-1961), Lewellen (1962) and Webb (1962) have investigated steady-state vortex flows that are axisymmetric. Of these, Lewellen's treatment is the most comprehensive. Gutman (1957) and Kuo (1964) have worked out solutions which incorporate the effects of thermal energy input on the flow. Boundary layer effects have been discussed by Rott (1962). Meteorological application has been made by Vaughn (1928), Humphreys (1940), Williams (1945), Battan (1958), as well as Kuo and Webb. Sinclair (1964) has made some measurements of temperature and velocity profiles in dust devils using a portable instrument probe.

The instability of rotating flows has been investigated by a number of people, including Harrison (1921), Taylor (1923), Synge (1938), Lin (1955) and Chandrasekhar (1962); but their studies have dealt primarily with

Couette flow. So far as the authors know, no comprehensive investigation of vortex stability independent of bounding walls has been carried out.

In this report, some significant results of these prior investigations are briefly presented and discussed, and where appropriate generalizations of these results are offered. A partial analysis of the stability of vortex flow is also included. Finally, a program of future investigations is outlined.

II. STEADY-STATE VORTEX FLOW

The starting point for all discussions of laminar viscous vortex flow is the set of equations of fluid motion (Navier-Stokes) expressed in cylindrical coordinates, e.g. Hinze (1959) pp 22-23:

$$\frac{\partial u}{\partial t} + Du - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right] \tag{2.1}$$

$$\frac{\partial v}{\partial t} + Dv + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right] \tag{2.2}$$

$$\frac{\partial w}{\partial t} + Dw = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\nabla^2 w \tag{2.3}$$

Here r, θ and z are the radial, angular and axial coordinates respectively; t is the time; u, v and w are the components of velocity in the r, θ and z directions respectively; and p is the pressure, ρ the density, ν the kinematic molecular viscosity and g the acceleration due to gravity. The operators D and ∇^2 are defined as

$$D \equiv u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

In the following discussions, except for the study of small perturbations, the motion is assumed to be axisymmetric, so that the derivatives with respect to θ vanish in the above equations, resulting in considerable simplification. Furthermore, in a first approach to understanding the motions in a "dust-devil," it is convenient to assume that the horizontal velocity components, u and v, do not vary in the vertical. This neglects the problem of flow in the surface boundary layer, but should be appropriate for the flow pattern at some distance above the surface. Equations (2.1), (2.2) and (2.3), using the above assumptions, become

$$u \frac{\partial u}{\partial r} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) - \frac{u}{r^2}\right] \qquad (2.4)$$

$$u \frac{\partial v}{\partial r} + \frac{uv}{r} = \nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v}{\partial r}\right) - \frac{v}{r^2}\right] \qquad (2.5)$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right] \qquad (2.6)$$

where $u = u(r)$, $v = v(r)$, $w = w(r,z)$, $p = p(r,z)$, ν = constant and $\rho = \rho(z)$ constant. It is assumed that the earth's rotation has little effect on the size of disturbance being considered, hence the Coriolis terms have been neglected. It is further assumed that the viscosity ν is constant; this assumption will be adequate for molecular viscosity, but is in doubt when eddy viscosity is considered.

The equation of mass continuity, assuming that the density is approximately constant, may be written as

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) = -\kappa(r) = -\frac{\partial w}{\partial z}, \text{ or } u = -\frac{1}{r}\int_0^r r\kappa(r)\,dr \qquad (2.7)$$

where $\kappa(r)$ is the horizontal convergence, a function of radius only. With the introduction of this convergence term, the flow may be considered two-dimensional in the r,θ plane. $\kappa(r)$ then plays the role of a distributed velocity sink.

The vertical component of vorticity, the only component present in this two-dimensional setup, is, in cylindrical coordinates,

$$\zeta = \frac{1}{r}\frac{\partial}{\partial r}(rv) \tag{2.8}$$

Substitution of (2.8) in (2.5) gives

$$u\zeta = \nu\frac{\partial\zeta}{\partial r} \tag{2.9}$$

as noted by Lamb (1932, p. 579). Integration yields

$$\zeta = \zeta_o \exp {}_o\!\int^r \frac{u(r)}{\nu}\,dr, \quad \zeta_o = \text{central vorticity} \tag{2.10}$$

and a second integration yields

$$rv = \zeta_o\, {}_o\!\int^r r \exp\left[{}_o\!\int^r \frac{u(r)}{\nu}\,dr\right] dr \tag{2.11}$$

For various special choices of $u(r)$, equation (2.11) may be solved analytically for rv and v. One such case is when

$$u = \frac{-\kappa_o r}{2} \tag{2.12}$$

Then (2.7) gives $\kappa(r) = \kappa_o$ = constant, stating that horizontal convergence is uniform for this case. Such an assumption may hold quite well over a large region near the central axis of the vortex. The resulting tangential velocity profile is

$$v = \frac{\Gamma_\infty}{2\pi r}\left[1 - \exp\left(\frac{-\kappa_o r^2}{4\nu}\right)\right] \tag{2.13}$$

where $\Gamma_\infty = \frac{4\pi\nu\zeta_o}{\kappa_o}$ is the circulation at $r \to \infty$. This is essentially the solution of Burgers (1940) and Rott (1958). If v_m is defined as the

maximum tangential velocity and r_m the corresponding radius at which the maximum occurs, then differentiation of (2.13) yields

$$a \equiv \frac{\kappa_o r_m^2}{2\nu} = 2.51 \qquad (2.14)$$

as a characteristic number for a vortex with this particular velocity profile, as implicitly stated by Rott. Furthermore,

$$\frac{\Gamma_m}{\Gamma_\infty} = \frac{a}{a+1} = .72, \qquad \Gamma_m \equiv 2\pi r_m v_m \qquad (2.15)$$

indicating the deviation of this velocity profile from the simple conservation of angular momentum, Γ = constant, one. If molecular viscosity of air, 0.14 cm^2 sec^{-1}, is used in (2.14), then, for a typical dust-devil dimension of r_m = 100 cm, the convergence κ_o is found to be 7×10^{-5} sec^{-1}. If the converging air starts at a radius of r_1 and moves inward to a radius r_2 in time t, this radial inflow, obtained by integrating (2.12) from r_1 to r_2, is given by

$$r_2/r_1 = e^{-\kappa_o t/2}, \quad \text{or} \quad t = \frac{2}{\kappa_o} \ln (r_1/r_2) \qquad (2.16)$$

For $r_1/r_2 = 2$, $t = 2 \times 10^4$ sec or about 5 hours, which is unrealistic. If the molecular viscosity is replaced by an effective eddy viscosity of, e.g., 10^2 cm^2 sec^{-1}, $\kappa_o = 5 \times 10^{-2}$ sec^{-1} for r_m = 100 cm, and for $r_1/r_2 = 2$, t = 28 sec, a reasonable period.

The eddy viscosity should be taken at least in the outer portion of the dust devil but probably not in the center part where solid rotation is approximated. Rott argues similarly for the application to a tornado.

The vorticity for the above case assumes a Gaussian distribution:

$$\zeta = \zeta_o e^{-\kappa_o r^2/4\nu} \qquad (2.17)$$

Another case of interest is that for which the convergence is zero outside of a "sink" region of radius r_1. For this case $u = -b/r$ for $r \geq r_1$, and the resulting angular momentum distribution is

$$rv = r_1 v_1 + \frac{\zeta_1 r_1^2}{\frac{b}{\nu} - 2}\left[1 - \left(\frac{r_1}{r}\right)^{\frac{b}{\nu} - 2}\right] \tag{2.18}$$

which is the solution obtained by Webb (1962) for the same sink assumption. For rv to be a bounded function for large r, it is required that $b/\nu \geq 2$. For $b/\nu = 2$, the inviscid solution $rv = r_1 v_1$ results. The ratio of the circulations Γ_∞ and $\Gamma_1 = 2\pi r_1 v_1$ is

$$\frac{\Gamma_1}{\Gamma_\infty} = \frac{\frac{b}{\nu} - 2}{\frac{b}{\nu} - 1} \tag{2.19}$$

The behavior of the flow within the sink region must be specified by some other condition, such as uniform convergence, etc.

A third analytic case of some interest is one in which u is a constant. This unrealistically implies infinite convergence and vertical acceleration along the central axis, but may serve as a limiting case for vortices where the convergence and updraft are highly concentrated near the center, in contrast to the uniform convergence case. With u = constant, the resulting velocity profile is

$$v = \frac{\Gamma_\infty}{2\pi r}\left[1 - e^{-ur/\nu}\left(1 + \frac{ur}{\nu}\right)\right] \quad \text{where } \Gamma_\infty = \frac{2\pi\zeta_0 \nu^2}{u^2} \tag{2.20}$$

and where, in the same manner as in the first solution (2.14) and (2.15),

$$\beta = \frac{ur_m}{\nu} = 1.80, \quad \frac{\Gamma_m}{\Gamma_\infty} = \frac{\beta^2}{\beta^2 + \beta + 1} = 0.54 \tag{2.21}$$

The velocity profiles given by (2.13) and (2.20) have very similar shapes despite the great contrast in the respective convergence patterns, see figure ( ).

Of course, these velocity profiles may be pieced together; for example, $u = \frac{-\kappa_o r}{2}$ for $0 \leq r < R$, $u = \frac{-\kappa_o R}{2}$ for $r \geq R$, then

$$rv = \frac{2\nu\zeta_o}{\kappa_o}\left(1 - e^{-\kappa_o R^2/4\nu}\right) + \frac{4\nu^2\zeta_o}{\kappa_o^2 R^2}\left[e^{-\kappa_o R^2/2\nu}\left(1 + \frac{\kappa_o R^2}{2\nu}\right) - e^{-\kappa_o Rr/2\nu}\left(1 + \frac{\kappa_o Rr}{2\nu}\right)\right] \tag{2.22}$$

for $r \geq R$, whereas equation (2.13) applies for $r < R$. Ultimately, the only justification for constructing such profiles is that they are readily obtainable in analytic form, which does not imply that they are very good approximations to reality.

A flow pattern which is somewhat more general than the above, and which presumably has greater correspondence with reality, is one in which the horizontal convergence is assumed to be a "hill-shaped" function,

$$\kappa(r) = \frac{a^2\kappa_o}{a^2 + r^2} \tag{2.23}$$

where a is the radius at which κ falls to $\kappa_o/2$. The corresponding function for radial velocity is then

$$u = -\frac{a^2\kappa_o}{2r}\ln\left(1 + \frac{r^2}{a^2}\right) \tag{2.24}$$

The maximum radial velocity is readily shown to occur at $r = 1.986\ a$ and has the value $u_m = 0.402\ \kappa_o a$. The tangential velocity cannot be obtained in analytical form for this case, but it can readily be computed numerically using finite-difference integration methods. For purposes of comparison, the following dimensionless quantities are defined:

$$x \equiv r/r_m, \quad y \equiv v/v_m, \quad \delta \equiv a/r_m, \quad \alpha \equiv \frac{\kappa_o r_m^2}{2\nu} \tag{2.25}$$

where v_m is the maximum tangential velocity and r_m the corresponding radius. The parameter δ gives a measure of the concentration of convergence near the central axis; for very small δ the convergence is highly concentrated along the axis, for $\delta \to \infty$ the convergence profile approaches uniformity. Results of the numerical calculations for κ/κ_o and y as functions of x are plotted in figure (2). The relation between α and δ is plotted in figure (3), showing that $\alpha \to 2.51$ as $\delta \to \infty$. It is noteworthy that for values of $\delta > 1$, the deviation of the tangential velocity from the uniform convergence solution (2.13) is very small.

More generally, the convergence can be expressed in the form of a series

$$\kappa(r) = \sum_{i=1}^{n} \frac{a_i \kappa_{o_i}}{a_i^2 + r^2} \tag{2.26}$$

where the a_i and κ_{o_i} are individually different values of half-width and central convergence. For $n = 2$, $a_1 < a_2$, $\kappa_{o1} < 0$ and $\kappa_{o_2} > 0$, a concentric two-cell vortex results, with divergence dominant in a central "eye" region and convergence elsewhere. Such a profile is possibly characteristic of the tornado. Gutman (1957), Sullivan (1959), Donaldson and Sullivan (1960) and Kuo (1964) have obtained solutions for concentric two-cell vortices. For all the foregoing solutions, the pressure profile is given very closely by

$$p = p_o + \rho \int_0^r \left(\frac{v^2}{r}\right) dr \tag{2.27}$$

where p_o is the central pressure, ρ the density for the height of interest and $v(r)$ the tangential velocity profile for the particular vortex. The other

terms of equation (2.4) upon integration give a small correction that results in reducing the pressure gradient given by (2.27) slightly.

The dissipation of energy in the vortex due to viscous forces has been discussed by Burgers (1948) and Rott (1959). Rott gives the energy equation for axisymmetric flow in the following form:

$$\rho u T \frac{dS}{dr} = \rho \nu \Phi + \frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) \tag{2.28}$$

where S is entrophy, T temperature, k thermal conductivity, and Φ the viscous dissipation function given by (Lamb, op. cit., p. 580)

$$\Phi = 2\left(\frac{\partial u}{\partial r}\right)^2 + 2\left(\frac{u}{r}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)^2 + \left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right)^2 \tag{2.29}$$

For Burgers' uniform convergence solution, the resulting values are

$$\Phi = \left(\frac{dv}{dr} - \frac{v}{r}\right)^2 + 6\kappa_o^2 \tag{2.31}$$

$$D \approx \frac{\rho \kappa_o \Gamma_\infty^2}{8\pi} \tag{2.32}$$

as given by Burgers. The approximate equality sign indicates here that the $6\kappa_o^2$ term is negligible in the integration of Φ out to a radius much larger than r_m. Thus the dissipation function is primarily a function of the tangential shear $(dv/dr - v/r)$ and the resulting dissipation depends on the "basic circulation," Γ_∞, as well as the convergence. It can be shown that the dominant dissipation term is always the tangential shear, and that (2.32) is a good approximation for the other vortices discussed above as well.

Equation (2.28) can now be used to calculate the dynamically induced temperature difference between the center of the vortex and the environment, as is done by Rott for a perfect gas with various thermal conductivities. For the limiting case of no conductivity, he finds

$$T_\infty - T_0 = \frac{1}{c_p} \frac{a\Gamma_\infty^2}{16\pi^2 r_m^2} \tag{2.33}$$

where T_0 is the central temperature and T_∞ is the temperature as $r \to \infty$. Hence the motion results in cooling as the air approaches the vortix center; the viscous dissipation that acts as a heat source is more than offset by the adiabatic expansion and cooling that results from the reduced central pressure. Using a value of $v_m = 10^3$ cm sec^{-1}, hence

$$\Gamma_\infty/2\pi r_m = (a + 1)\, v_m/a = 1.4 \times 10^3,$$

one finds that $T_\infty - T_0 = 0.12$ degrees C. Rott shows that for the conductivity of air, a temperature deficit of roughly 1/2 this magnitude is correct.

Sinclair's (1964) measurements show a temperature rise of from 3 to 9 degrees near the center of dust devils at a height of approximately 2 meters. It thus appears that the dynamic reduction of temperature is negligible compared to the effects of the heated surface boundary layer of air, which is presumably advected into the core of the vortex.

III. NON-STEADY STATES OF THE VISCOUS VORTEX

Vortex With Uniform Convergence

The decay of the vortex with uniform convergence appears to have first been obtained by Terazawa (1922, 1923) together with that for a model with v independent of r and a combination "vortex" with v independent of r in the core and v proportional to 1/r outside the core.

The following treatment differs only slightly from that of Terazawa, who worked in terms of vorticity rather than tangential velocities.

By setting the convergence equal to zero and retaining $\partial v/\partial t$, the tangential equation of motion (2.2) becomes

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2} + \frac{1}{x}\frac{\partial y}{\partial x} - \frac{y}{x^2} \tag{3.1}$$

after transforming to the non-dimensional variables

$$x = r(\tau)/r_m(o), \qquad y = v(\tau)/v_m(o), \qquad \tau = \nu t/r_m^2(o)$$

Integration by separation of variables gives

$$y = \int_o^\infty c_\lambda \, J_1 \, (\lambda^{1/2} x) \, e^{-\lambda\tau} d\lambda \tag{3.2}$$

where c_λ remains to be determined.

From the initial conditions

$$(y)_{\tau=o} = \frac{\alpha + 1}{\alpha} (1 - e^{-\alpha x^2/2}) \tag{3.3}$$

and with the use of the Hankel integral transform

$$c_\lambda = \frac{\alpha + 1}{2\alpha} \int_o^\infty J_1(\lambda^{1/2} x) \, (1 - e^{-\alpha x^2/2}) \, dx \tag{3.4}$$

giving

$$y = \frac{\alpha + 1}{2\alpha} \int_{\lambda=0}^{\infty} \left\{ \int_{x=0}^{\infty} J_1(\lambda^{1/2}x)\left[1 - e^{-\alpha x^2/2}\right]dx \right\} J_1(\lambda^{1/2}x)\, e^{-\lambda\tau} d\lambda \quad (3.5)$$

The indicated integration may be carried out by means of confluent hypergeometric functions to give

$$y = \frac{\alpha + 1}{\alpha x} \left[1 - \exp\left\{ - \frac{\alpha x^2}{2} \left(\frac{1}{1 + 2\alpha\tau}\right)\right\}\right] \quad (3.6)$$

This result has been plotted for various values of τ in figure (4).

The limiting form of the solution for the decay of the vortex given by (3.6) as τ becomes very large may be obtained easily from similarity considerations.* If the decay time is taken to be great enough that the solution has become independent of the initial conditions, dimensional analysis leads to a solution of the form.

$$v = \nu f(\eta)/r \qquad \text{where} \quad \eta = r^2/\nu t \quad (3.7)$$

This allows the partial differential equation (3.1) to be expressed as an ordinary differential equation

$$4f''(\eta) + f'(\eta) = 0 \quad (3.8)$$

which has the solution

$$v = \frac{\Gamma_0}{2\pi r}\left(1 - e^{-r^2/4\nu t}\right) \quad (3.9)$$

or in dimensionless form

$$y = \frac{\alpha + 1}{\alpha x}\left(1 - e^{-\frac{x^2}{4\tau}}\right) \quad (3.9a)$$

This equation is seen to be the same as the limit of equation (3.6) for $\tau \to \infty$.

The location of the radius of maximum tangential velocity as a function of the decay time may be obtained by differentiating (3.9a) with respect to x giving

*This appears to have been done first by Taylor (1918).

$$\frac{dy}{dx} = \frac{\alpha + 1}{\alpha x^2}\left[1 - e^{-x^2/4\tau}\right] + \frac{\alpha + 1}{\alpha x}\left(\frac{2x}{4\tau}\right) e^{-x^2/4\tau} \tag{3.10}$$

so, since $\frac{dy}{dx} = 0$ at y_m, it follows that

$$\frac{x_m^2}{2\tau} + 1 = e^{x_m^2/4\tau}$$

This has the solution $\frac{x_m^2}{2\tau} = 2.51$ which is identical to α from equation (2.14).

Thus

$$y_m = \frac{\alpha + 1}{\alpha x_m}\left(1 - e^{-x_m^2/4\tau}\right) = \frac{\alpha + 1}{\alpha x_m}\left(1 - e^{-2\alpha}\right)$$

Replacing $e^{\alpha/2}$ by $(1 + \alpha)$ and multiplying by x_m gives

$$x_m y_m = \frac{\alpha + 1}{\alpha}\left(1 - \frac{1}{1 + \alpha}\right) = 1 \tag{3.11}$$

Recalling, for the constant convergence model, from (3.6)

$$y = \frac{\alpha + 1}{\alpha x}\left(1 - e^{-\alpha x^2/2(1 + \alpha\tau)}\right)$$

It is noted that this equation has the same form as (3.9a) with a change of variable given by

$$\tau' = \frac{1}{2\alpha} + \tau \tag{3.12}$$

so that the locus of the maxima for (3.6) is also given by (3.11).

<u>Vortex With "Hill Function" Convergence</u>

The decay of the vortex generated by the "hill function" distribution of convergence is obtained by numerical methods from (3.1) using as initial conditions the numerical solution of

$$\frac{d^2y}{dx^2} + \frac{1}{x}\left[\alpha\delta^2 \ln\left(1 + x^2/\delta^2\right) + 1\right]\frac{dy}{dx} + \left[\alpha\delta^2 \ln\left(1 + x^2/\delta^2\right) - 1\right] y = 0 \qquad (3.13)$$

where

$$\delta = a/r_m, \quad \alpha = \frac{\kappa_o r_m^2}{2\nu} = \frac{\kappa_o a^2}{2\nu\delta^2}, \quad x = r/r_m, \quad y = v/v_m$$

Rewriting (3.1) as a finite difference equation

$$\frac{y_{i,j+1} - y_{i,j}}{h_t} = \frac{\nu}{r_m^2}\left[\frac{y_{i+1,j} - 2y_{i,j} + y_{i-1,j}}{h_x^2} + \frac{1}{x_{i,j}}\left(\frac{y_{i+1,j} - y_{i-1,j}}{2h_x}\right) - \frac{y_{i,j}}{x_{i,j}^2}\right] \qquad (3.14)$$

and taking for boundary conditions

$$y = 0 \text{ at } x = 0 \text{ for all } t, \text{ and } y_{i+1} \equiv y_i \text{ at } x = 4.95$$

permits the numerical solution to be carried out until t becomes large enough that the similarity solution becomes valid. Stability of the finite difference solution requires, according to Hildebrand (1952), p. 232, that

$$h_t \leq 1/2 \, \frac{h_x^2 \, r_m^2(o)}{\nu} \qquad (3.15)$$

Figure (5) indicates the results for an example with moderately high concentrations of convergence near the center, corresponding to the point $\delta = 0.24$. Here the "hill function" velocity has been normalized to give the same value of angular momentum at infinity as the non-dimensional "Burgers' model." It is noted that the locus of the maximum tangential velocity rapidly approaches that for the similarity solution as τ becomes large.

Rouse's Model

Rouse (1963) has presented a vortex model produced by a rotating cylinder in a viscous fluid, by considering the cylinder to shrink to an

infinitely small radius, while simultaneously increasing in rotational speed, to maintain a constant peripheral circulation Γ_g. After accelerating the surrounding fluid by the transmission of shear for a finite generation time, t_g, the generating cylinder is abruply brought to rest, and the surrounding fluid allowed to decelerate. After a decay time of t_d the distribution of tangential velocity is given by

$$Y = \exp\left(\frac{-X}{1+T}\right) - \exp\left(\frac{-X}{T}\right) \qquad (3.16)$$

where

$$X = r^2/4\nu t_g, \quad Y = 2\pi r v/\Gamma_g, \quad T = t_d/t_g \qquad (3.17)$$

Here a different product $X_m Y_m$ may be shown to approach the limit $1/e$ as $T \to \infty$, so that here the value of the maximum circulation becomes, in the limit, inversely proportional to the square of the radius.*

Since the solution of (3.17) satisfies the partial differential equation (3.1), the similarity solution (3.9) should be approached as a limit. Rewriting it in terms of X, Y and T, the similarity solution becomes

$$Y = 1 - \exp\left(\frac{-X}{T}\right) \quad \text{if } t \leq t_d \qquad (3.18)$$

In this case the similarity solution is approached only for $T \ll 1$, that is for a generating time t_g very large in relation to the decay time t_d.

Vortex Growth

Rott (1958) has discussed the growth of a vortex to show how the steady state solution may be approached from any initial distribution.

*Note that in Burgers' model, equation (2.13), the maximum value of circulation remains at infinite radius.

For the viscous vortex

$$\frac{\partial\Gamma}{\partial t} - \frac{\kappa r}{2}\frac{\partial\Gamma}{\partial r} = \nu\left[\frac{\partial^2\Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial\Gamma}{\partial r}\right] \tag{3.19}$$

assumes a solution of the form

$$\Gamma = \Gamma\ (\sigma) \qquad \text{where } \sigma = rF(t) \tag{3.20}$$

then (3.19) becomes

$$r\dot{F}\Gamma' - \frac{\kappa}{2} rF\Gamma' = \nu\left[F^2\Gamma'' - \frac{1}{r}F\Gamma'\right] \tag{3.21}$$

where

$$\dot{F} = \frac{dF}{dt} \quad \text{and} \quad \Gamma' = \frac{d\Gamma}{d\sigma}$$

Rearranging (3.21) to give

$$(\dot{F} - \frac{\kappa}{2} F)\ \sigma\Gamma' = \nu F^3\left[\Gamma'' - \frac{1}{\sigma}\Gamma'\right] \tag{3.22}$$

and choosing

$$\dot{F} - \frac{\kappa}{2} F = - \frac{C}{2} F^3 \tag{3.23}$$

where C is some constant, (3.22) assumes the form of the steady state equation which has been shown to have the solution

$$\Gamma = \Gamma_\infty\ (1 - e^{-C\sigma^2/4\nu}) \tag{3.24}$$

Integrating (3.23) gives

$$F = \left(\frac{C}{\kappa} + Ae^{-\kappa t}\right)^{-1/2} \tag{3.25}$$

where A is the constant of integration.

Substituting r^2F^2 for σ^2, (3.24) becomes

$$\Gamma = \Gamma_\infty\left[1 - \exp\ - \frac{\kappa r^2}{4\nu}\cdot\frac{1}{1 + \frac{\kappa A}{C}e^{-\kappa t}}\right] \tag{3.26}$$

as $t \to \infty$ the steady state solution.*

*For $\kappa = 0$, $F^2 = C/t$, (3.24) becomes the similarity solution (3.9).

For $t = 0$ the initial circulation is

$$\Gamma_i = \Gamma_\infty \left[1 - \exp\left(\frac{-\kappa r^2}{4\nu} \cdot \frac{1}{1 + \kappa B}\right)\right] \tag{3.27}$$

where $B = A/C$.

Since any initial distribution of Γ may be synthesized by the summation of solutions (3.27) with varying values of B, all initial distributions of Γ must tend to the steady state form, provided κ is constant.

If κ is not constant but is a function of time, then (3.23) becomes

$$\frac{dF}{dt} - \frac{\kappa(t)}{2} F + \frac{C}{2} F^3 = 0$$

Letting $F^2 = G$, then

$$\frac{dG}{dt} - \kappa(t)G + CG^2 = 0$$

Now letting $G = PQ$

$$\frac{1}{Q}\frac{dQ}{dt} + \left\{\frac{1}{P}\frac{dP}{dt} - \kappa(t)\right\} + CPQ = 0$$

Choosing P so that $\frac{1}{P}\frac{dP}{dt} - \kappa(t) = 0$

$$P = e^{\int\kappa(t)dt}$$

and

$$\frac{1}{Q}\frac{dQ}{dt} + CQe^{\int\kappa(t)dt} = 0$$

so

$$-\frac{1}{Q} + C\int e^{\int\kappa(t)dt}dt = 0$$

then

$$Q = \frac{1}{C\int e^{\int\kappa(t)dt}dt}$$

and

$$F^2 = G = \frac{e^{\int\kappa(t)dt}}{C\int e^{\int\kappa(t)dt}dt} \tag{3.28}$$

It is of interest to look at $\kappa(t)$ when it has the form of $D\cos\alpha t$ indicating alternate convergence and divergence. Now .

$$F^2 = \frac{e^{\sin\alpha t}}{E\int e^{\sin\alpha t}dt}$$

where E is a constant.

It is readily seen that $e^{\sin\alpha t}$ oscillates between e^{-1} and e, so that $E\int e^{\sin\alpha t}dt$ increases monotonically with t, and F^2 oscillates between decreasing limits but approaches zero as $t \to \infty$, so that it follows from (3.24) that $\Gamma \to 0$ also.

IV. STABILITY OF VORTEX FLOW

The classical approach of linearized equations for small perturbations in the mean flow, as discussed by J. L. Synge (1938), C. C. Lin (1955) and others, may be used to determine criteria for hydrodynamic stability of the vortex motion. In the general case, the superimposed perturbation is assumed to be free to travel in any direction whatever and to have its amplitude vector oriented at random, requiring a complicated mathematical formulation. More simply, the perturbed motion may be considered to consist of several component traveling waves--nine in all--of which three are compression waves and the others transverse waves. The mathematical analysis, tentative as yet owing to lack of observational confirmation, is outlined below for one of the perturbation modes--that of a sinusiodal wave traveling circularly and having its amplitude vector in the r,θ plane.

The motion is assumed to consist of a mean component plus a perturbation component, so that

$$u_* = \bar{u} + u, \quad v_* = \bar{v} + v, \quad p_* = \bar{p} + p \tag{4.1}$$

where u_* is the <u>total</u> radial velocity, $\bar{u}$ the mean value, and u the perturbation component. (A notation more consistent with what has gone before would be $u = \bar{u} + u'$, where u' is the perturbation quantity, but is not adopted here in order to avoid having primed quantities in the following equations.) Refer to equations (2.1) and (2.2), where u is now replaced with $u_* = \bar{u} + u$, etc. The mean values, $\bar{u}$, $\bar{v}$ and $\bar{p}$, are seen to satisfy the equations seperately; and so these terms may be removed from the equations, leaving equations of motion for the perturbation. As it is assumed that $u \ll \bar{u}$, etc., the non-linear terms may be neglected. The resulting equations for the motion of small horizontal perturbations are then

$$\frac{\partial u}{\partial t} + \frac{\bar{v}}{r}\frac{\partial u}{\partial \theta} - \frac{2\bar{v}v}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right] \tag{4.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial \bar{v}}{\partial r} + \frac{\bar{v}}{r}\frac{\partial v}{\partial \theta} + \frac{u\bar{v}}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\right] \tag{4.3}$$

The perturbed motion must satisfy the continuity condition

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \theta}\right] = 0 \tag{4.4}$$

Equation (4.4) permits a stream function $\psi(r,\theta,t)$ to be defined such that

$$u \equiv \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v \equiv -\frac{\partial \psi}{\partial r} \tag{4.5}$$

If equation (4.2) is operated on by $(\frac{1}{r}\frac{\partial}{\partial \theta})$ and equation (4.3) by $(-\frac{1}{r} - \frac{\partial}{\partial r})$, and the equations then added, the pressure is eliminated from the resulting equation, which, by use of (4.5), may be written

$$\psi_{rrt} + \frac{1}{r^2}\psi_{\theta\theta t} + \frac{1}{r}\psi_{rt} = \nu\psi_{rrrr} + \frac{2\nu}{r^2}\psi_{rr\theta\theta} + \frac{\nu}{r^4}\psi_{\theta\theta\theta\theta} + \frac{2\nu}{r}\psi_{rrr} - \frac{\bar{v}}{r}\psi_{rr\theta}$$

$$- \frac{2\nu}{r^3}\psi_{r\theta\theta} - \frac{\bar{v}}{r^3}\psi_{\theta\theta\theta} - \frac{\nu}{r^2}\psi_{rr} - \frac{\bar{v}}{r^2}\psi_{r\theta} + \frac{4\nu}{r^4}\psi_{\theta\theta} + \frac{\nu}{r^3}\psi_r + \left[\frac{\bar{v}_{rr}}{r} + \frac{\bar{v}_r}{r^2} - \frac{\bar{v}}{r^3}\right]\psi_\theta \tag{4.6}$$

where the suffixes indicate partial derivatives.

Assuming a solution for a traveling, circular wave,

$$\psi = f(r)e^{in(\theta - \omega t)} \tag{4.7}$$

where the amplitude $f(r)$ and the angular velocity $\omega(r)$ may be complex, but the wave number n is a real integer, equation (4.6) transforms into

$$f'''' + \frac{2}{r}f''' - \frac{(1 + 2n^2)}{r^2} f'' + \frac{(1 + 2n^2)}{r^3} f' + \frac{n^2(n^2 - 4)}{r^4} f$$

$$= \frac{in}{\nu}\left\{\left(\frac{\bar{v}}{r} - \omega\right) f'' + \frac{1}{r}\left(\frac{\bar{v}}{r} - \omega\right) f' + \frac{1}{r^2}\left[-n^2\left(\frac{\bar{v}}{r} - \omega\right) + \frac{\bar{v}}{r} - \bar{v}_r - r\bar{v}_{rr}\right] f\right\} \tag{4.8}$$

which is strictly appropriate only for time $t = 0$.

For purposes of calculation, it is desirable to introduce the following dimensionless quantities in terms of $\bar{v}_m$, the maximum mean tangential velocity, and r_m, the corresponding radius:

$x \equiv \dfrac{r}{r_m}$ (dimensionless radius)

$\bar{y} \equiv \dfrac{\bar{v}}{\bar{v}_m}$ (dimensionless mean tangential velocity)

$\phi(x) \equiv \dfrac{1}{r_m \bar{v}_m} f(r)$ (dimensionless perturbation amplitude)

$\bar{\sigma} \equiv \dfrac{r_m \bar{v}}{r \bar{v}_m} = \dfrac{\bar{y}}{x}$ (dimensionless angular velocity of mean flow)

$$\sigma \equiv \frac{r_m \omega}{\bar{v}_m} \qquad \text{(dimensionless angular velocity of perturbations)}$$

$$R \equiv \frac{\bar{v}_m r_m}{\nu} \qquad \text{(characteristic Reynolds Number for vortex)}$$

Using these definitions, equation (4.8) may be expressed in dimensionless form as

$$\phi'''' + \frac{2}{x}\phi''' - \frac{(1 + 2n^2)}{x^2}\phi'' + \frac{(1 + 2n^2)}{x^3}\phi' + \frac{n^2(n^2 - 4)}{x^4}\phi$$

$$= inR\left[(\bar{\sigma} - \sigma)\left(\phi'' + \frac{1}{x}\phi' - \frac{n^2}{x^2}\phi\right) - \frac{1}{x^2}\left(\frac{\bar{y}}{x} - \bar{y}' - x\bar{y}''\right)\phi\right] \qquad (4.9)$$

where the primes now indicate differentiation with respect to x.

Returning to a consideration of the streamfunction for horizontal vortex flow; since for mean motions $\bar{u} \ll \bar{v}$, a total streamfunction for v may be approximately expressed as

$$\begin{aligned} \psi_s &= \bar{\psi} + \psi \\ &= \bar{\psi}(r) + \delta\bar{\psi}(r)e^{in(\theta - \omega t)} \\ &= \bar{\psi}(r) + \frac{\partial\bar{\psi}}{\partial r}\,\delta r e^{in(\theta - \omega t)} \\ &= \bar{\psi}(r) + \bar{v}\delta\, r e^{in(\theta - \omega t)} \end{aligned} \qquad (4.10)$$

Comparison of (4.7) and (4.10) indicates that $f(r) = \bar{v}(r)\delta r$, and since δr may be regarded as an arbitrarily small displacement, it follows that $f(r) = \bar{v}(r)$ or, in dimensionless notation,

$$\phi(x) = \bar{y}(x) = x\bar{\sigma}(x) \qquad (4.11)$$

Replacing $\phi(x)$ by $\bar{y}$ in equation (4.9) and expressing the perturbation angular velocity as the sum of real and imaginary components, $\sigma = \sigma_r + i\sigma_i$, leads to the following expressions

$$\frac{\sigma_r}{\sigma} = 1 + \frac{x^2\bar{y}'' + x\bar{y}' - \bar{y}}{x^2\bar{y}'' + x\bar{y}' - n^2\bar{y}} \tag{4.12}$$

$$R\sigma_i = \frac{\bar{y}'''' + \frac{2}{x}\bar{y}''' - \frac{(1+2n^2)}{x^2}\bar{y}'' + \frac{(1+2n^2)}{x^3}\bar{y}' + \frac{n^2(n^2-4)}{x^4}\bar{y}}{n(\bar{y}'' + \frac{1}{x}\bar{y}' - \frac{n^2}{x^2}\bar{y})} \tag{4.13}$$

where σ_r is the angular phase speed of the perturbations and σ_i is the amplification factor, positive for initially unstable perturbations.

Burgers' solution for the vortex with uniform horizontal convergence, equation (2.13), can be written in dimensionless notation as

$$\bar{y} = \frac{\alpha + 1}{\alpha x}\left(1 - e^{-\alpha x^2/2}\right) \tag{4.14}$$

where $\alpha \equiv \kappa r_m^2/2\nu = 2.51$. Curves for $R\sigma_i$ for various wave numbers n, using this expression for $\bar{y}$, are plotted in figure (6). These indicate great stability for $n > 1$ in the central core region, where the mean flow approaches solid rotation. The analysis also indicates that perturbations of wave numbers 1 and 2 are initially unstable for sufficiently large radii; qualitatively, wave number 1 perturbations do appear to grow in the outer parts of vortices, but quantitative data is at present lacking.

To summarize, the above analysis pertains to a rather special case of perturbed motion, where the only perturbed motion permitted is one with radial amplitude and angular movement. Of course, a truly "general" perturbation which is free to move in any direction and whose amplitude function is three-dimensional may have an entirely different stability criterion, probably highly dependent upon direction of propogation. Thus, the overall question of stability is only partially answered by the treatment above;

however, determination of stability criteria for the remaining modes of perturbation will result in a 3 x 3 matrix of stability criteria that should also describe the general perturbation.

V. PLANS FOR FURTHER STUDIES

Three main avenues of study are indicated:

1. Extension of the theory to three-dimensional flow, incorporating the effects of convection resulting from various distributions of surface heating and including the effects of friction in the surface boundary layer.

2. Continuation of the analysis of instability for other perturbation modes, attempting to construct a complete picture of dynamic instability associated with various vortex models.

3. Development of apparatus for the purpose of conducting studies of models representative of the smaller scale intense vortices, in particular dust devils and fire-induced whirlwinds.

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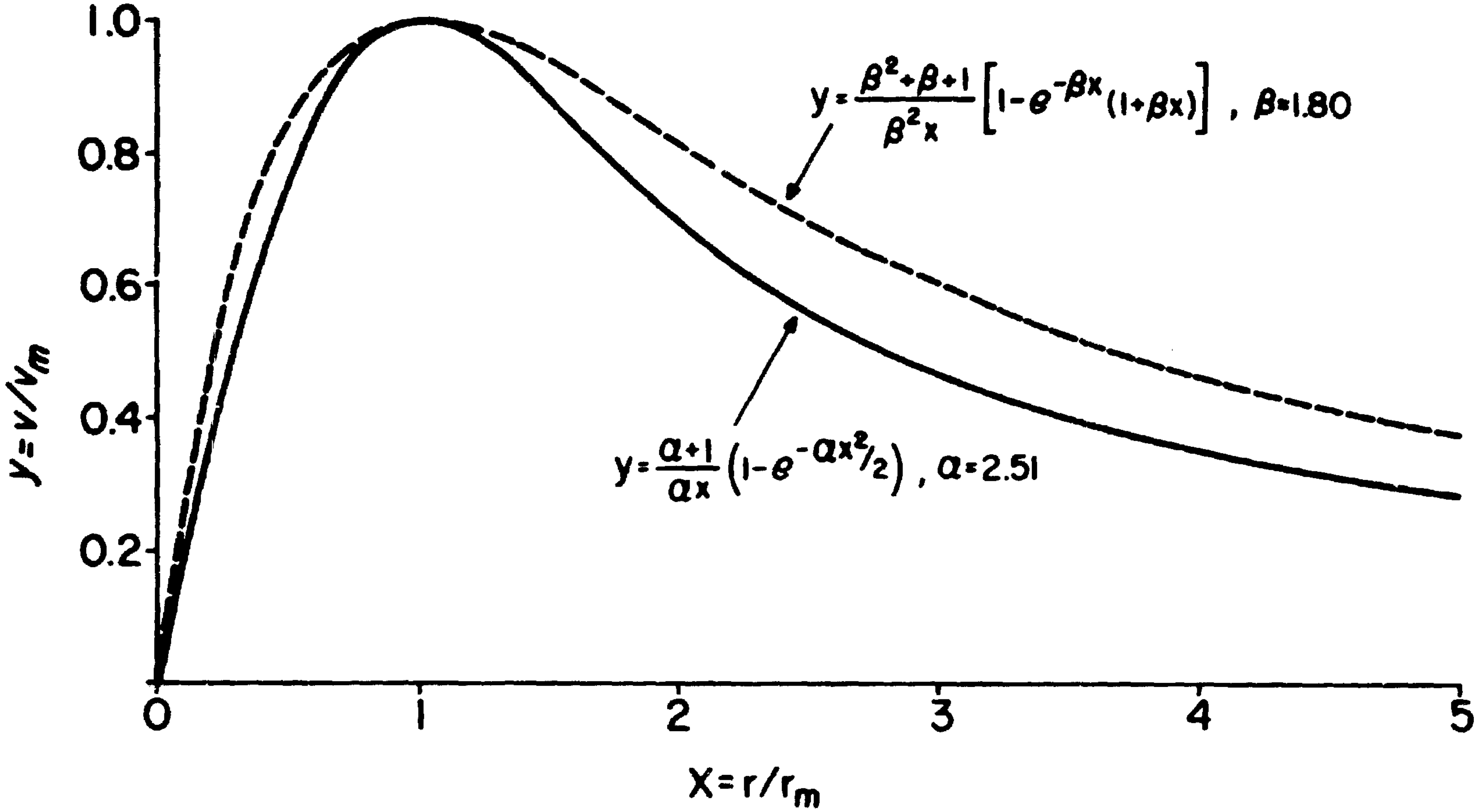

FIG. 1: DIMENSIONLESS TANGENTIAL VELOCITY FOR VORTEX WITH UNIFORM HORIZONTAL CONVERGENCE, EQUATION (2.13), COMPARED TO VORTEX WITH HORIZONTAL CONVERGENCE INVERSELY PROPORTIONAL TO RADIUS, EQUATION (2.20).

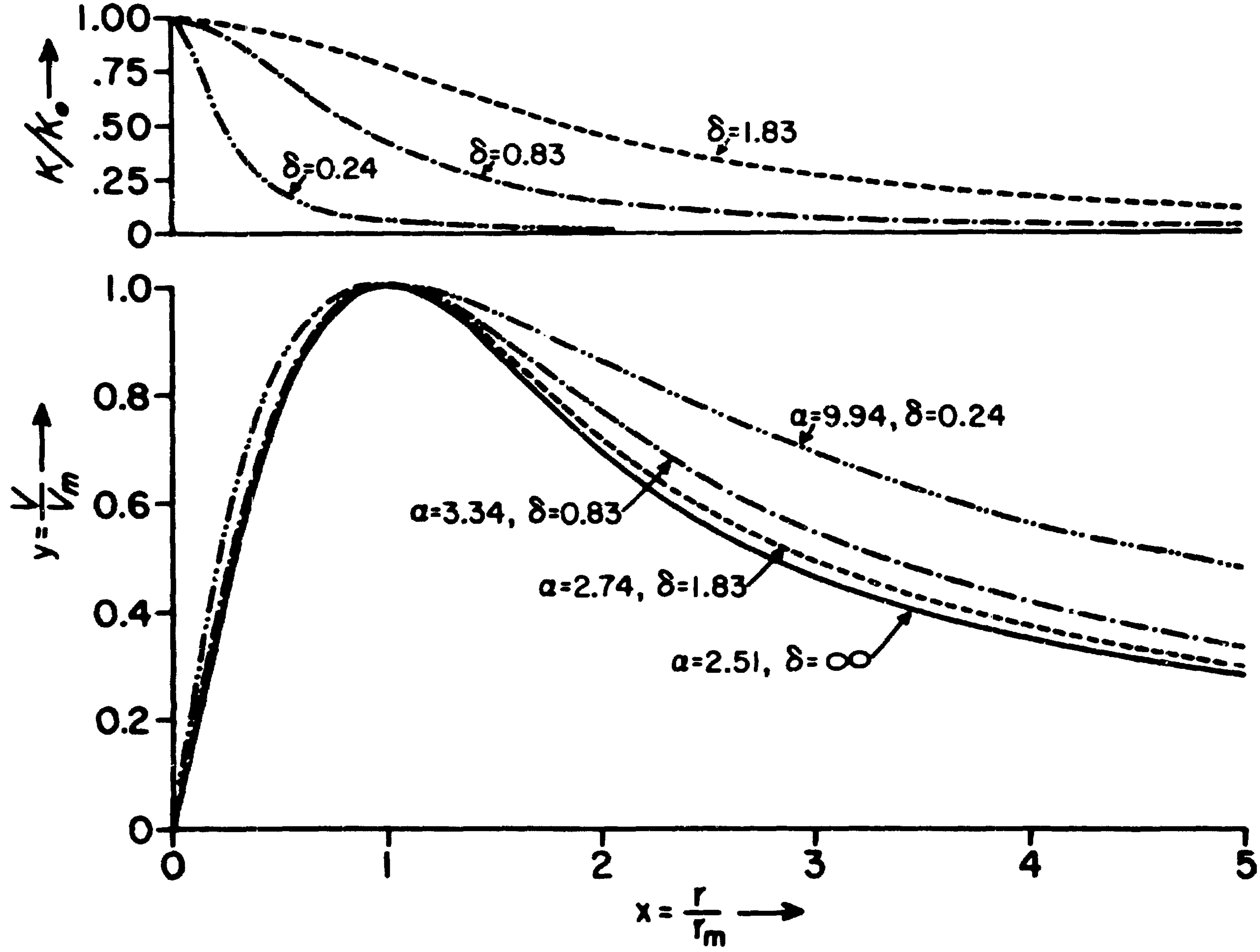

FIG. 3: DIMENSIONLESS TANGENTIAL VELOCITY FOR VARIOUS "HILL FUNCTION" CONVERGENCE PROFILES.

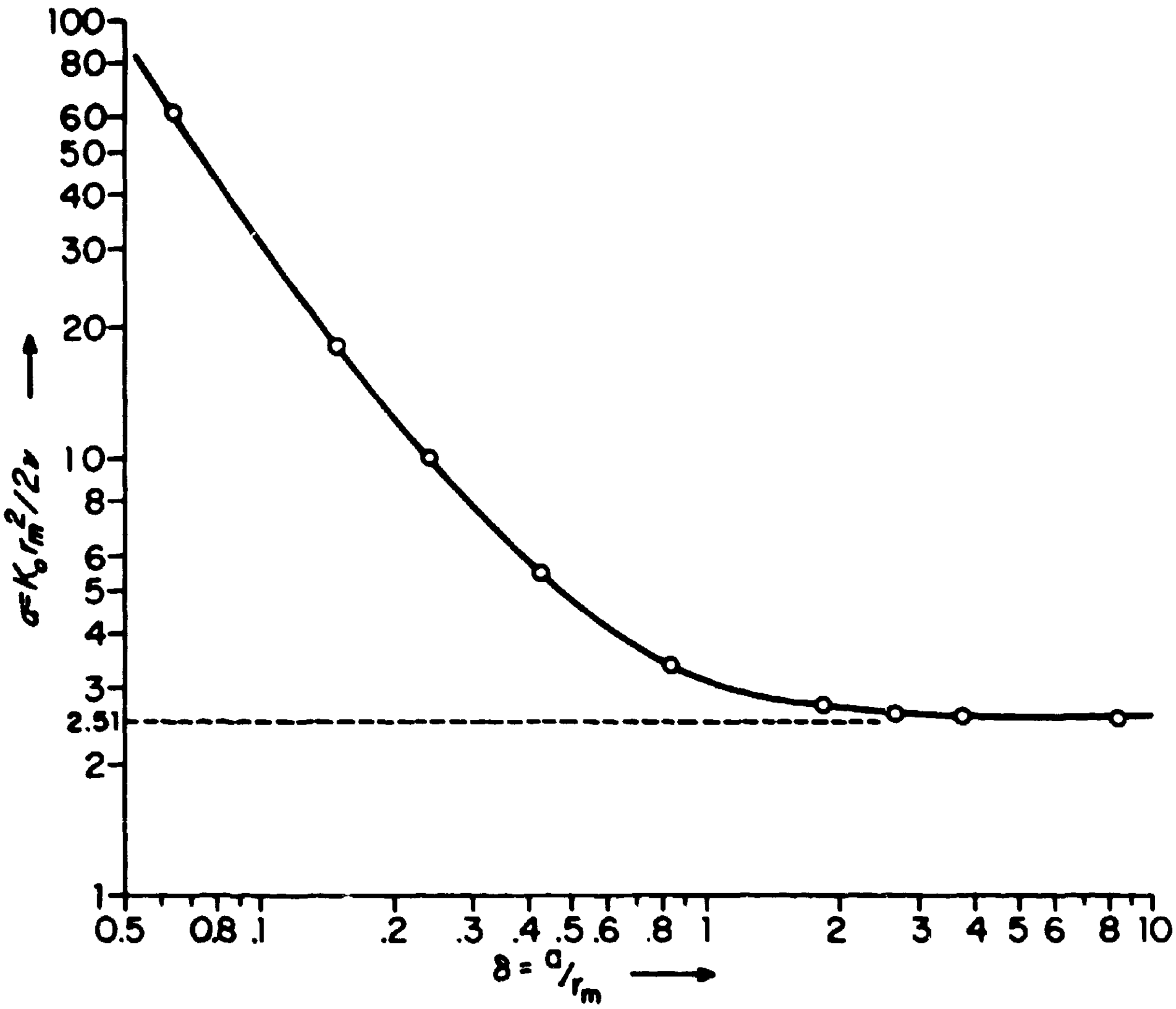

FIG. 3: RELATION BETWEEN CHARACTERISTIC CONVERGENCE PARAMETER σ AND DIMENSIONLESS "HILL FUNCTION" HALF-WIDTH δ.

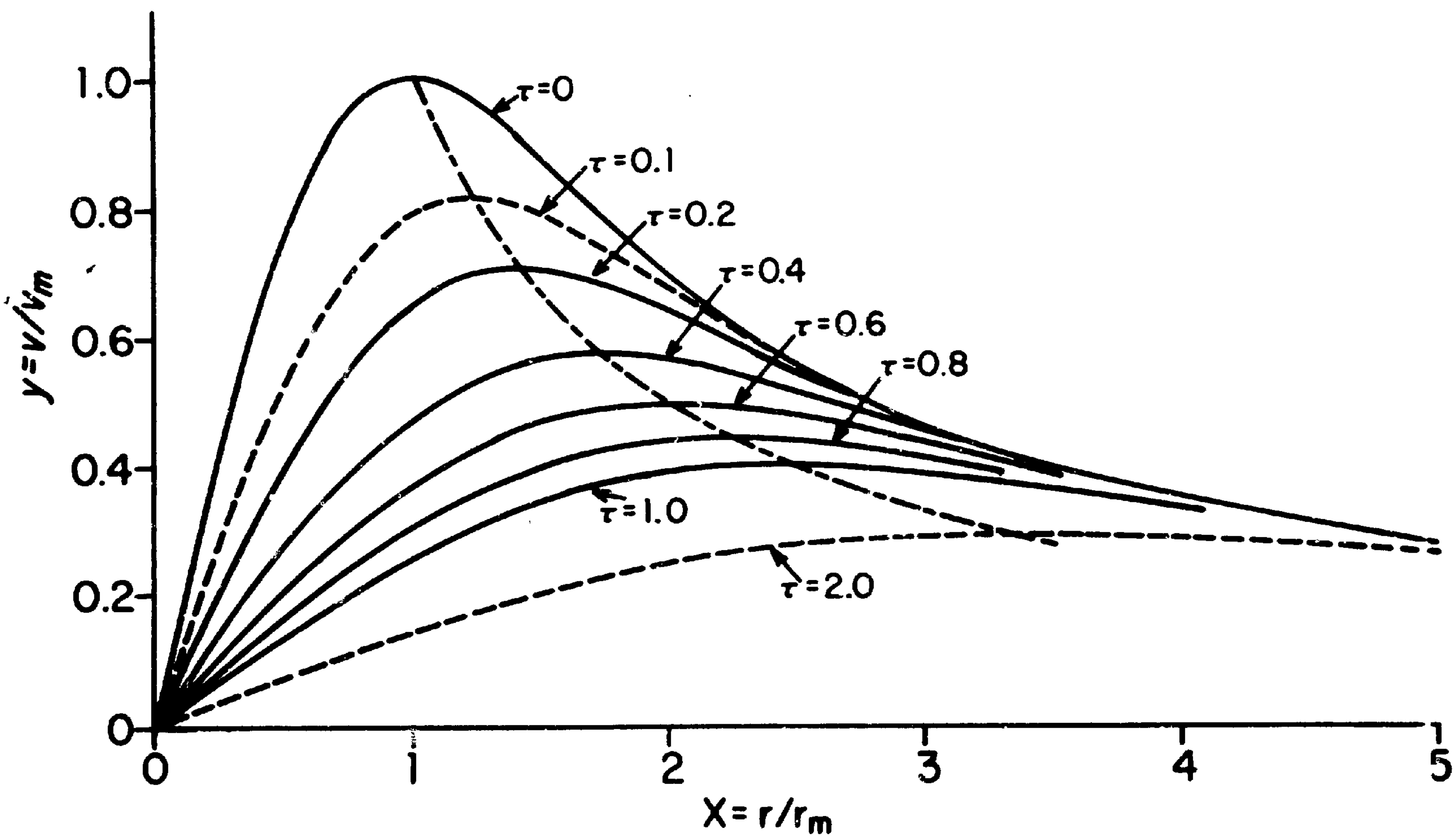

FIG. 4: DECAY OF THE UNIFORM HORIZONTAL CONVERGENCE VORTEX AS A FUNCTION OF DIMENSIONLESS TIME $\tau = \nu t^2/r_m^2(o)$.

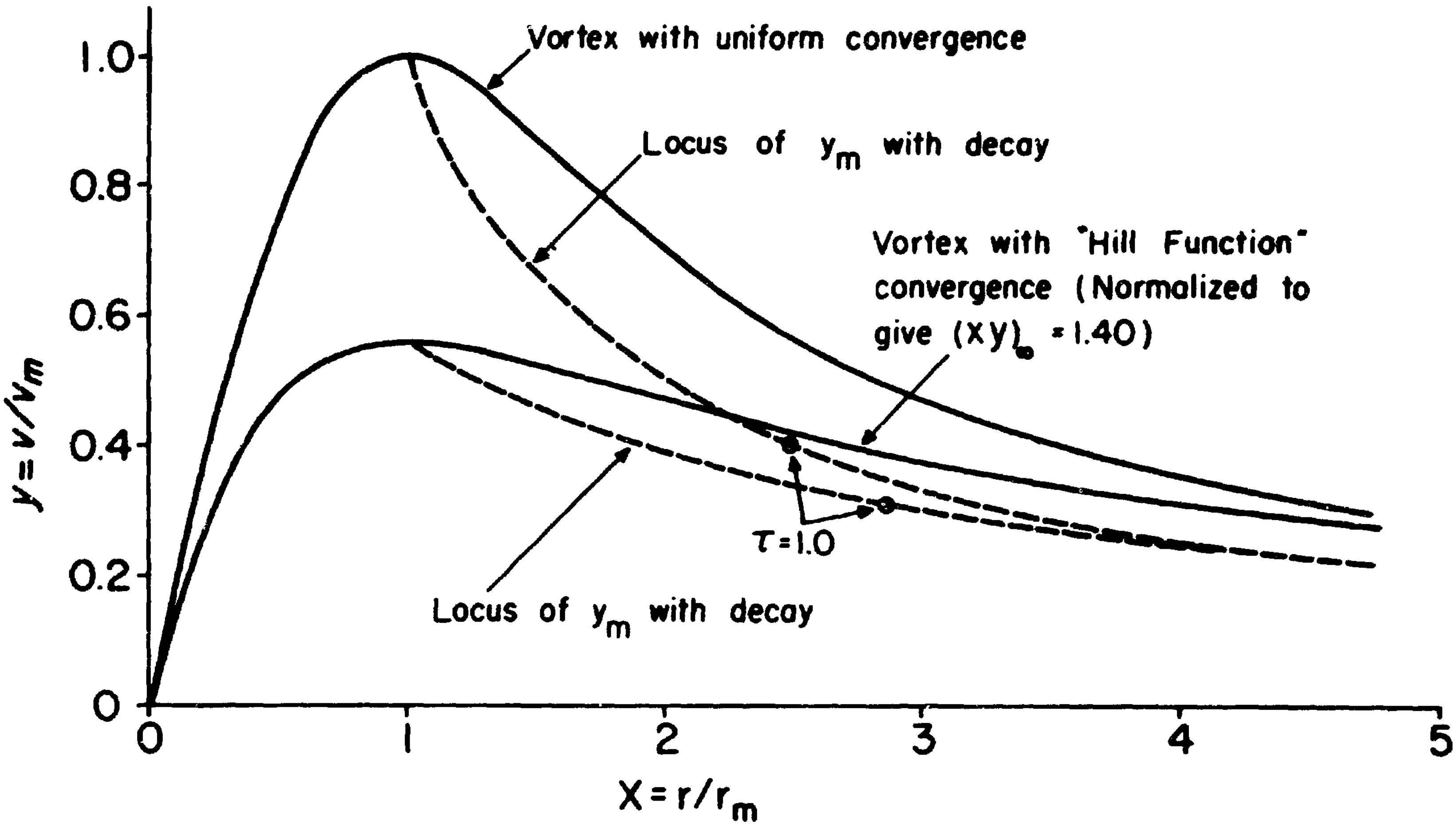

FIG. 5: COMPARISON OF "HILL FUNCTION" AND UNIFORM HORIZONTAL CONVERGENCE VORTEX DECAYS.

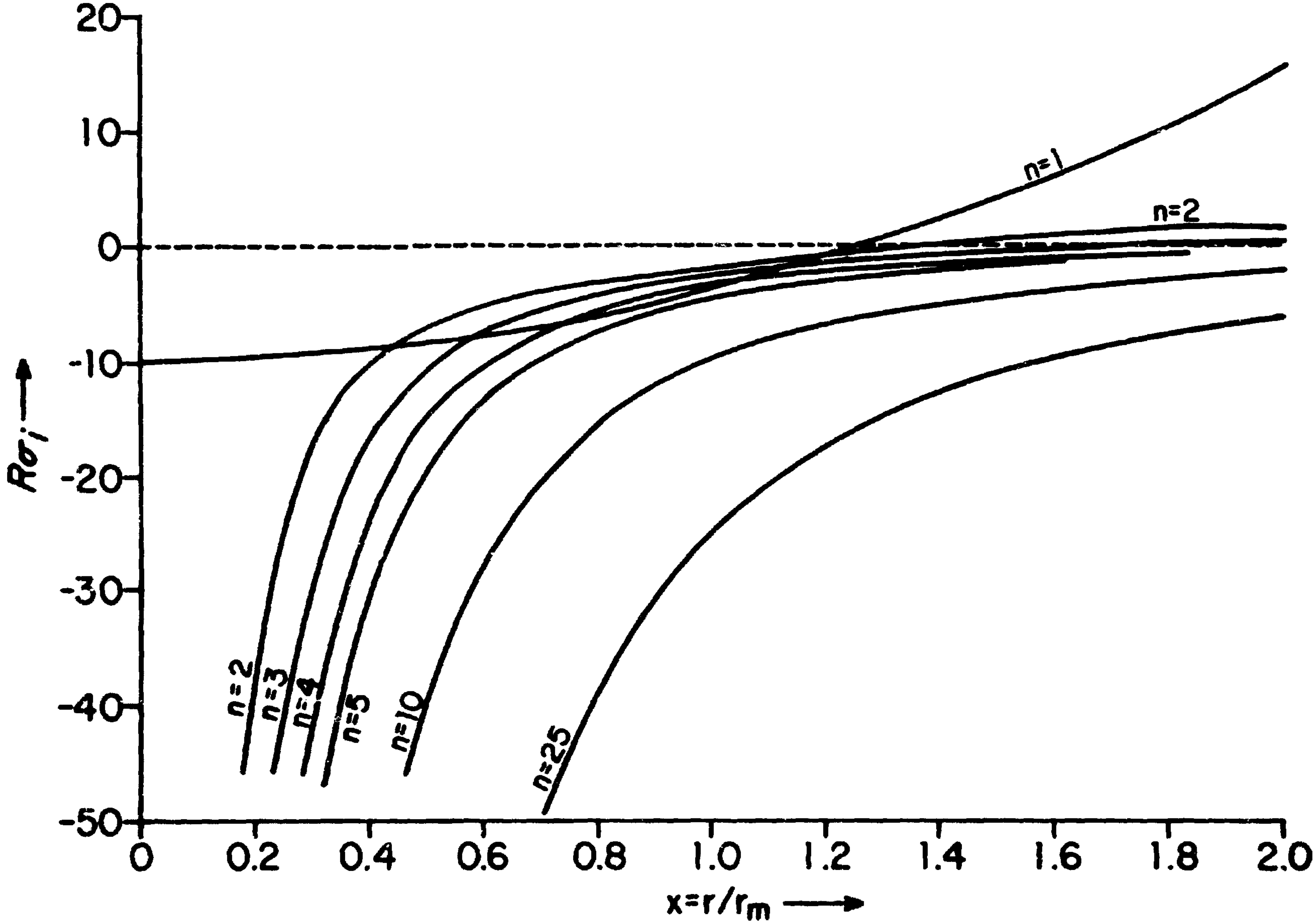

FIG. 6: DIMENSIONLESS INITIAL GROWTH RATES OF PERTURBATIONS OF WAVE NUMBER n FOR THE UNIFORM HORIZONTAL CONVERGENCE VORTEX.